Metodou konečných prvků vyřešte okrajový problém:

$$-0.5y'' + 2y' = x \quad \text{pro } x \in (0;1)$$
$$y(0) = 1$$
$$0.5y'(1) = 2y(1) - 1$$

$h = 0.25$

ODVOZENÍ SLABÉ FORMULACE

$W = \{y;\ y \in H^1((0;1)) \text{ a } y(0) = 1\}$ ... množina řešení

$V = \{v;\ v \in H^1((0;1)) \text{ a } v(0) = 0\}$ ... prostor testovacích funkcí

$$-0.5y''v + 2y'v = xv \quad ; \forall v \in V$$

$$-0.5\int_0^1 y''v\,dx + 2\int_0^1 y'v\,dx = \int_0^1 xv\,dx \quad ; \forall v \in V$$

$$-0.5\left([y'v]_0^1 - \int_0^1 y'v'\,dx\right) + 2\int_0^1 y'v\,dx = \int_0^1 xv\,dx \quad ; \forall v \in V$$

$$-\underbrace{0.5y'(1)}_{=2y(1)-1}v(1) + 0.5y'(0)\underbrace{v(0)}_{=0} + 0.5\int_0^1 y'v'\,dx + 2\int_0^1 y'v\,dx = \int_0^1 xv\,dx \quad ; \forall v \in V$$

$$(-2y(1) + 1)v(1) + 0.5\int_0^1 y'v'\,dx + 2\int_0^1 y'v\,dx = \int_0^1 xv\,dx \quad ; \forall v \in V$$

$$-2y(1)v(1) + 0.5\int_0^1 y'v'\,dx + 2\int_0^1 y'v'\,dx = -v(1) + \int_0^1 xv\,dx$$

SLABÁ FORMULACE

Najděte $y \in W = \{y;\ y \in H^1((0;1)) \text{ a } y(0) = 1\}$ takové, že pro něj platí

$$\mathcal{A}(y,v) = \mathcal{L}(v) \quad ; \forall v \in V,$$

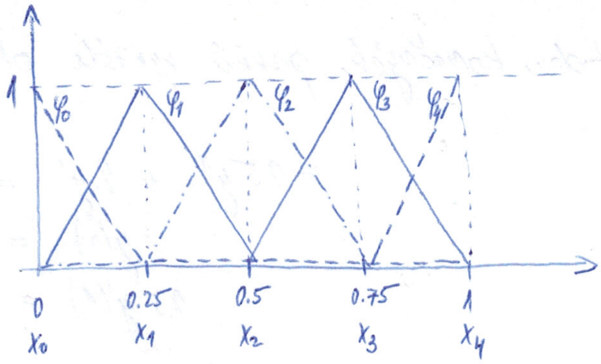
kde

$\mathcal{A}(y,v) = 0.5\int_0^1 y'v'\,dx + 2\int_0^1 y'v'\,dx - 2y(1)v(1)$, ... bilineární forma

$\mathcal{L}(v) = \int_0^1 xv\,dx - v(1)$, ... lineární funkcionál

$V = \{v;\ v \in H^1((0;1)) \text{ a } v(0) = 0\}$.

SÍŤ: $x_0 = 0$; $x_1 = 0.25$; $x_2 = 0.5$;
$x_3 = 0.75$; $x_4 = 1$



PŘIBLIŽNÉ ŘEŠENÍ:

$$y_h = \sum_{j=0}^{4} y_j \varphi_j = 1 \cdot \varphi_0 + y_1 \varphi_1 + y_2 \varphi_2 + y_3 \varphi_3 + y_4 \varphi_4$$

$\Rightarrow$ počet neznámých : 4 $\quad y_1, y_2, y_3, y_4$

za testovací funkce volím : $\varphi_1, \varphi_2, \varphi_3, \varphi_4$

$$\varphi_0 = \begin{cases} \frac{0-x}{0.25} & x \in \langle 0 ; 0.25 \rangle \\ 0 & \text{jindy} \end{cases} \qquad \varphi_1 = \begin{cases} \frac{x-0}{0.25} & x \in \langle 0 ; 0.25 \rangle \\ \frac{0.5-x}{0.25} & x \in \langle 0.25 ; 0.5 \rangle \\ 0 & \text{jindy} \end{cases}$$

$$\varphi_2 = \begin{cases} \frac{x-0.25}{0.25} & x \in \langle 0.25 ; 0.5 \rangle \\ \frac{0.75-x}{0.25} & x \in \langle 0.5 ; 0.75 \rangle \\ 0 & \text{jindy} \end{cases} \qquad \varphi_3 = \begin{cases} \frac{x-0.5}{0.25} & x \in \langle 0.5 ; 0.75 \rangle \\ \frac{1 - x}{0.25} & x \in \langle 0.75 ; 1 \rangle \\ 0 & \text{jindy} \end{cases}$$

$$\varphi_4 = \begin{cases} \frac{x-0.75}{0.25} & x \in \langle 0.75 ; 1 \rangle \\ 0 & \text{jindy} \end{cases}$$

SOUSTAVA LINEÁRNÍCH ROVNIC

$$\begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{array} & \begin{bmatrix} \mathcal{A}(\varphi_1,\varphi_1) & \mathcal{A}(\varphi_2,\varphi_1) & \mathcal{A}(\varphi_3,\varphi_1) & \mathcal{A}(\varphi_4,\varphi_1) \\ \mathcal{A}(\varphi_1,\varphi_2) & \mathcal{A}(\varphi_2,\varphi_2) & \mathcal{A}(\varphi_3,\varphi_2) & \mathcal{A}(\varphi_4,\varphi_1) \\ \mathcal{A}(\varphi_1,\varphi_3) & \mathcal{A}(\varphi_2,\varphi_3) & \mathcal{A}(\varphi_3,\varphi_3) & \mathcal{A}(\varphi_4,\varphi_3) \\ \mathcal{A}(\varphi_1,\varphi_4) & \mathcal{A}(\varphi_2,\varphi_4) & \mathcal{A}(\varphi_3,\varphi_4) & \mathcal{A}(\varphi_4,\varphi_4) \end{bmatrix} \end{array} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \mathcal{L}(\varphi_1) - \mathcal{A}(1 \cdot \varphi_0, \varphi_1) \\ \mathcal{L}(\varphi_2) \\ \mathcal{L}(\varphi_3) \\ \mathcal{L}(\varphi_4) \end{bmatrix}$$

$$\underset{K}{\shortparallel} \qquad\qquad \underset{y}{\shortparallel} \qquad\qquad \underset{F}{\shortparallel}$$

$\hookrightarrow$ matice tuhosti $\qquad$ vektor zatížení

- $K_{11} = \mathcal{A}(\varphi_1, \varphi_1) = 0.5\int_0^1 \varphi_1'\varphi_1' dx + 2\int_0^1 \varphi_1'\varphi_1 \, dx - 2\,\varphi_1(1)\varphi_1(1) =$

$$= 0.5\left(\int_0^{0.25} \left(+\frac{1}{0.25}\right)^2 dx + \int_{0.25}^{0.5} \left(-\frac{1}{0.25}\right)^2 dx \right) + 2\cdot\left(\int_0^{0.25} \frac{1}{0.25}\cdot\left(\frac{x}{0.25}\right) dx + \right.$$

$$\left. + \int_{0.25}^{0.5} \left(-\frac{1}{0.25}\right)\cdot\left(\frac{0.\cancel{2}5-x}{0.25}\right) dx \right) - 2\cdot 0\cdot 0$$

$$= 0.5(4+4) + 2\cdot\left(4\cdot\frac{1\cdot 0.25}{2} - 4\cdot\frac{1\cdot 0.25}{2}\right) - 0 = \underline{\underline{4}}$$

- $K_{22} = K_{33} = K_{11}$

- $K_{44} = \mathcal{A}(\varphi_4, \varphi_4) = 0.5\int_0^1 (\varphi_4')^2 dx + 2\int_0^1 \varphi_4'\varphi_4 - 2\,\varphi_4(1)\varphi_4(1) =$

$$= 0.5\int_{0.75}^1 \left(\frac{1}{0.25}\right)^2 dx + 2\cdot\frac{1}{0.25}\cdot\int_{0.75}^1 \frac{x-0.75}{0.25} dx - 2\cdot 1\cdot 1 =$$

$$= 0.5\cdot 4 + 2\cdot 4\cdot\frac{1\cdot 0.25}{2} - 2 = \underline{\underline{1}}$$

- $K_{12} = \mathcal{A}(\varphi_2, \varphi_1) = 0.5\int_0^1 \varphi_2'\varphi_1' dx + 2\int_0^1 \varphi_2'\varphi_1 \, dx - 2\,\varphi_2(1)\varphi_1(1)$

$$= 0.5\int_{0.25}^{0.5} \left(-\frac{1}{0.25}\right)\left(\frac{1}{0.25}\right) dx + 2\int_{0.25}^{0.5} \left(+\frac{1}{0.25}\right)\left(\frac{x-0.25}{0.25}\right) dx - 2\cdot 0\cdot 0$$

$$= 0.5\cdot(-4) + 2\cdot(+4)\cdot\frac{1\cdot 0.25}{2} - 0 = \underline{\underline{-1}}$$

- $K_{23} = K_{34} = K_{12}$

- $K_{21} = \mathcal{A}(\varphi_1, \varphi_2) = 0.5\int_0^1 \varphi_1'\varphi_2' dx + 2\int_0^1 \varphi_1'\varphi_2 \, dx - 2\,\varphi_1(1)\varphi_2(1)$

$$= 0.5\int_{0.25}^{0.5} \left(-\frac{1}{0.25}\right)\left(\frac{1}{0.25}\right) dx + 2\int_{0.25}^{0.5} \left(-\frac{1}{0.25}\right)\left(\frac{x-0.25}{0.25}\right) dx - 2\cdot 0\cdot 0 =$$

$$= 0.5(-4) + 2\cdot(-4)\cdot\frac{1\cdot 0.25}{2} - 0 = \underline{\underline{-3}}$$

- $K_{32} = K_{43} = K_{21}$

• $F_1 = \mathcal{L}(\varphi_1) - \mathcal{A}(1\cdot\varphi_0; \varphi_1)$

$$\mathcal{L}(\varphi_1) = \int_0^1 x\varphi_1 \, dx - \varphi_1(1) = \int_0^{0.25} x \frac{x}{0.25} dx + \int_{0.25}^{0.5} x \cdot \frac{0.5-x}{0.25} dx - 0$$

$$= \frac{1}{0.25}\left[\frac{x^3}{3}\right]_0^{0.25} + \frac{1}{0.25}\left[\frac{0.5x^2}{2} - \frac{x^3}{3}\right]_{0.25}^{0.5} = 0.0625$$

$$\mathcal{A}(1\cdot\varphi_0, \varphi_1) = 0.5\int_0^1 \varphi_0'\varphi_1' dx + 2\int_0^1 \varphi_0'\varphi_1 - 2\varphi_0(1)\varphi_1(1)$$

$$= 0.5\int_0^{0.25}\left(-\frac{1}{0.25}\right)\cdot\left(\frac{1}{0.25}\right) dx + 2\int_0^{0.25}\left(-\frac{1}{0.25}\right)\cdot\frac{x}{0.25} dx - 2\cdot 0\cdot 0 =$$

$$= 0.5\cdot(-4) + 2\cdot(-4)\cdot\frac{1\cdot 0.25}{2} - 0 = \underline{\underline{-3}}$$

$$F_1 = 0.0625 - (-3) = \underline{3{,}0625}$$

• $F_2 = \mathcal{L}(\varphi_2) = \int_0^1 x\varphi_2 \, dx - \varphi_2(1) = \int_{0.25}^{0.5} x \cdot \frac{x-0.25}{0.25} dx + \int_{0.5}^{0.75} x \frac{0.75-x}{0.25} dx - 0 = 0.125$

• $F_3 = \mathcal{L}(\varphi_3) = \int_0^1 x\varphi_3 \, dx - \varphi_3(1) = \int_{0.5}^{0.75} x \cdot \frac{x-0.5}{0.25} dx + \int_{0.75}^{1} x \cdot \frac{1-x}{0.25} dx - 0 = \dots = 0.1875$

• $F_4 = \mathcal{L}(\varphi_4) = \int_0^1 x\varphi_4 \, dx - \varphi_4(1) = \int_{0.75}^{1} x \cdot \frac{x-0.75}{0.25} dx - 1 = \dots = -0.8854$

$$\Downarrow$$

$$Ky = F$$

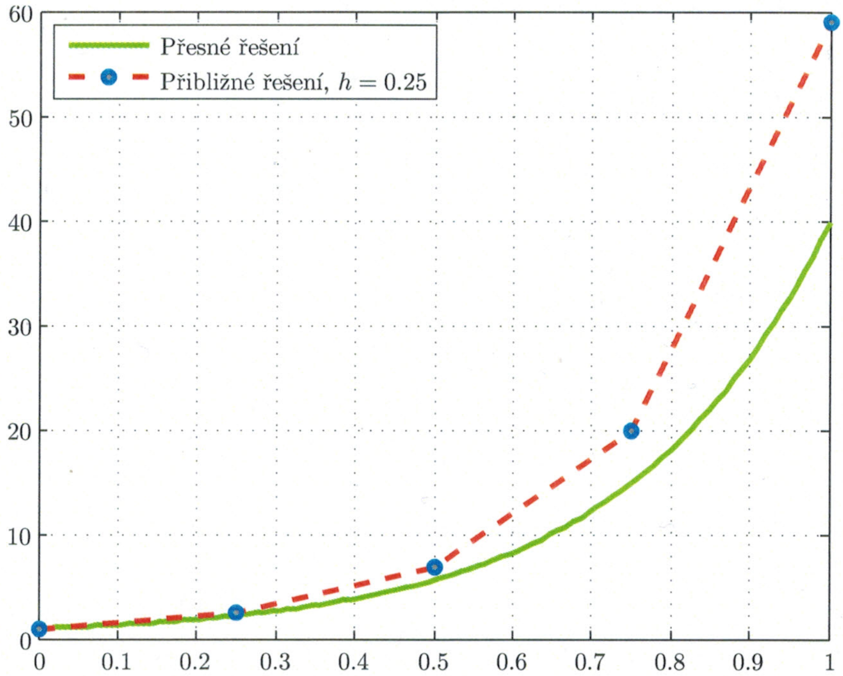
$$\begin{bmatrix} 4 & -1 & 0 & 0 \\ -3 & 4 & -1 & 0 \\ 0 & -3 & 4 & -1 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 3{,}0625 \\ 0.125 \\ 0.1875 \\ -0.8854 \end{bmatrix}$$

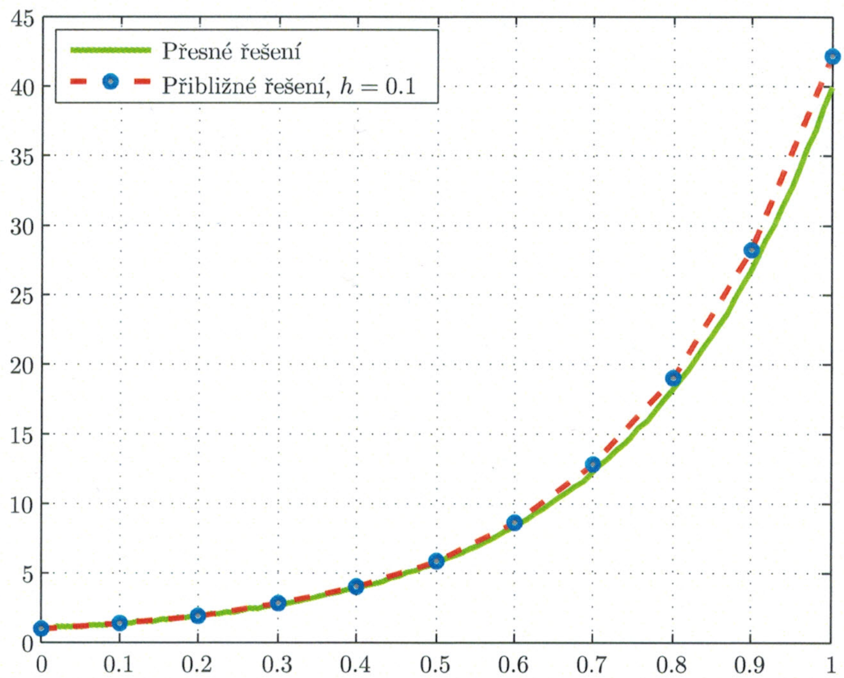
Řešení soustavy je:

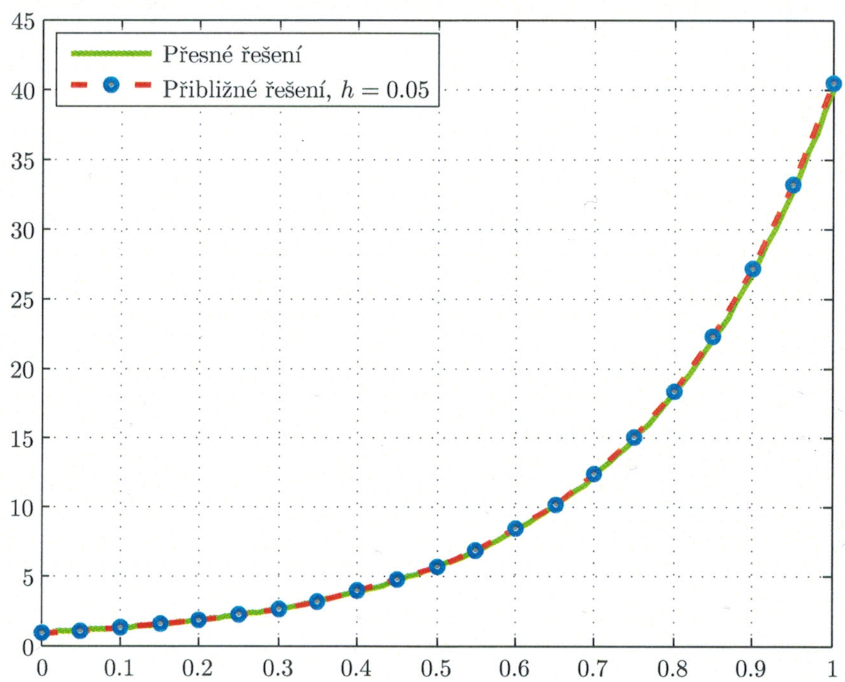
$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2{,}4896 \\ 6{,}8958 \\ 19{,}9896 \\ 56{,}0833 \end{bmatrix}$$

Konečné prvkové řešení diferenciální rovnice je:

$$y_h = \varphi_0 + 2.4896\,\varphi_1 + 6.8958\,\varphi_2 + 19.9896\,\varphi_3 + 56.0833\,\varphi_4 .$$


Přesné řešení
Přibližné řešení, $h = 0.25$
0
10
20
30
40
50
60
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Přesné řešení
Přibližné řešení, $h = 0.1$
0
5
10
15
20
25
30
35
40
45
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Přesné řešení
Přibližné řešení, $h = 0.05$
0
5
10
15
20
25
30
35
40
45
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1