Př

$$-2y'' + 0{,}2\,y = e^x \quad \text{na } (0,1)$$
$$y(0) = 2$$
$$y(1) = 3 \qquad h = 0{,}25$$

$V = \{ v \in H^1(0,1) \ ; \ v(0) = 0 \ \text{ a } \ v(1) = 0 \}$

$$-2y''v + 0.2yv = e^x v \qquad \forall v \in V$$

$$-2\int_0^1 y''v\,dx + 0.2\int_0^1 yv\,dx = \int_0^1 e^x v\,dx \qquad \forall v \in V$$

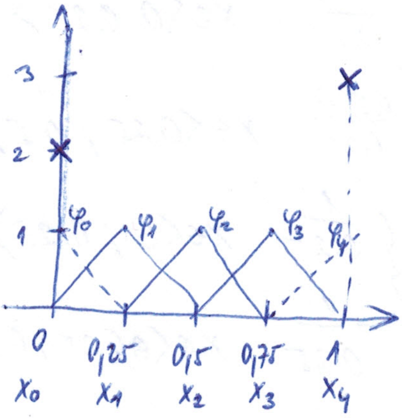
$$-2\left[y'v\right]_0^1 + 2\int_0^1 y'v'\,dx + 0.2\int_0^1 yv\,dx = \int_0^1 e^x v\,dx$$

$$-2\cdot\big(y'(1)\underbrace{v(1)}_{=0} - y'(0)\underbrace{v(0)}_{=0}\big) + 2\int_0^1 y'v'\,dx + 0.2\int_0^1 yv\,dx = \int_0^1 e^x v\,dx$$

$$2\int_0^1 y'v'\,dx + 0.2\int_0^1 yv\,dx = \int_0^1 e^x v\,dx$$

$W = \{ w \in H^1(0,1) \ ; \ w(0) = 2 \ \text{ a } \ w(1) = 3 \}$

slabá formulace : najděte $y \in W$ tak, aby
$$\mathcal{A}(y,v) = L(v) \qquad \forall\, v \in V,$$
kde
$$\mathcal{A}(y,v) = 2\int_0^1 y'v'\,dx + 0.2\int_0^1 yv\,dx$$
$$L(v) = \int_0^1 e^x v\,dx$$
$$V = \{ v \in H^1 \ ; \ v(0) = 0 \ \text{ a } \ v(1) = 0 \}$$
$$W = \{ v \in H^1 \ ; \ v(0) = 2 \ \text{ a } \ v(1) = 3 \}$$



Síť: x_0, x_1, x_2, x_3, x_4

Přibližné řešení:

$$y_h = \sum_{j=0}^{4} c_j \varphi_j =$$

$$= 2\varphi_0 + c_1\varphi_1 + c_2\varphi_2 + c_3\varphi_3 + 3\varphi_4$$

fce y_h je zvolena tak, aby náležela prostoru W.

Počet neznámých: 3

tj c_1, c_2, c_3

$\Rightarrow$ z prostoru V vyberu 3 fce v (tzv. testovací fce)

Výhodná je volba $\varphi_1, \varphi_2, \varphi_3$

- Vytvoření soustavy lin. rovnic $Kx = b$ $\quad x = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$K_{3\times3} = \begin{bmatrix} A(\varphi_1,\varphi_1) & A(\varphi_2,\varphi_1) & A(\varphi_3,\varphi_1) \\ A(\varphi_1,\varphi_2) & A(\varphi_2,\varphi_2) & A(\varphi_3,\varphi_2) \\ A(\varphi_1,\varphi_3) & A(\varphi_2,\varphi_3) & A(\varphi_3,\varphi_3) \end{bmatrix}$$

$$b_{3\times1} = \begin{bmatrix} L(\varphi_1) - A(2\varphi_0,\varphi_1) \\ L(\varphi_2) \\ L(\varphi_3) - A(3\varphi_4,\varphi_3) \end{bmatrix}$$

tyto členy tu jsou proto že v levém a pravém krajním bodě jsou zadány podmínky bez derivace

• $\mathcal{A}(\varphi_1,\varphi_1) = 2\int_0^1 \varphi_1'\varphi_1'\,dx + 0{,}2\int_0^1 \varphi_1\varphi_1\,dx$

$= 2\cdot\left[\int_0^{0.25}\left(\frac{1}{0.25}\right)^2 dx + \int_{0.25}^{0.5}\left(-\frac{1}{0.25}\right)^2 dx\right] +$

$+ 0{,}2\cdot\left[\int_0^{0.25}\left(\frac{x}{0.25}\right)^2 dx + \int_{0{,}25}^{0.5}\left(2-\frac{x}{0.25}\right)^2 dx\right.$

$= 16{,}033$

$\varphi_1 = \begin{cases} \frac{x-0}{0{,}25} & x\in\langle 0, 0.25\rangle \\ \frac{0{,}5-x}{0{,}25} & x\in\langle 0.25\,;\,0.5\rangle \end{cases}$

$\varphi_1' = \begin{cases} \frac{1}{0.25} & x\in\langle 0\,;\,0.25\rangle \\ -\frac{1}{0.25} & x\in\langle 0.25\,;\,0.5\rangle \end{cases}$

Je zřejmé že výrazy $\mathcal{A}(\varphi_2,\varphi_2)$ a $\mathcal{A}(\varphi_3,\varphi_3)$ vedou na stejný výsledek, protože představují stejnou plochu pod grafem.

• $\mathcal{A}(\varphi_1,\varphi_2) = \mathcal{A}(\varphi_2,\varphi_1) = 2\int_0^1 \varphi_2'\varphi_1'\,dx + 0{,}2\int_0^1 \varphi_2\varphi_1\,dx$

$= 2\int_{0.25}^{0.5}\frac{1}{0.25}\cdot\left(-\frac{1}{0.25}\right)dx + 0.2\int_{0.25}^{0.5}\frac{x-0.25}{0.25}\cdot\frac{0.5-x}{0.25}\,dx = -7{,}9917$

Výrazy $\mathcal{A}(\varphi_3,\varphi_2)$, $\mathcal{A}(\varphi_2,\varphi_3)$ vyjdou stejně

• $\mathcal{A}(\varphi_1,\varphi_3) = \mathcal{A}(\varphi_3,\varphi_1) = 2\int_0^1 \varphi_3'\varphi_1'\,dx + 0.2\int_0^1 \varphi_3\varphi_1\,dx = 0$

~~Dosazení~~

• $L(\varphi_1) = \int_0^1 e^x\varphi_1\,dx = \int_0^{0.25} e^x\cdot\frac{x}{0.25}\,dx + \int_{0.25}^{0.5} e^x\left(\frac{0.5-x}{0.25}\right)dx =$

$=$ Per-partes .... $= 0{,}1479 + 0.1748 = 0.3227$

$\mathcal{A}(2\varphi_0,\varphi_1) = 2\int_0^1 2\varphi_0'\varphi_1'\,dx + 0.2\int_0^1 2\varphi_0\varphi_1\,dx = 4\int_0 \left(-\frac{1}{0.25}\right)\left(\frac{1}{0.25}\right) +$

$+ 0.4\int_0^{0.25}\frac{0.25-x}{0.25}\,dx = -15{,}9833$

$b(1) = L(\varphi_1) - \mathcal{A}(2\varphi_0,\varphi_1) = 0.3227 + 15{,}9833 = 16{,}3060$

• $b(2) = L(\varphi_2) = \int_0^1 e^x\varphi_2\,dx = \int_{0.25}^{0.5} e^x\left(\frac{x-0.25}{0.25}\right)dx + \int_{0.5}^{0.75} e^x\left(\frac{0.75-x}{0.25}\right)dx$

$=$ per-partes $= 0.4143$

• $b(3) = L(\varphi_3) - \mathcal{A}(3\varphi_4,\varphi_3) = \int_0^1 e^x\varphi_3\,dx - \left[2\int_0^1 3\varphi_4'\varphi_3'\,dx + 0.2\int_0^1 3\varphi_4\varphi_3\,dx\right]$

$= 24{,}5070$

$$Kx = b$$

$$\begin{bmatrix} 16{,}0333 & -7{,}9917 & 0 \\ -7{,}9917 & 16{,}0333 & -7{,}9917 \\ 0 & -7{,}9917 & 16{,}0333 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 16{,}3060 \\ 0{,}4143 \\ 24{,}5070 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2{,}2996 \\ 2{,}5732 \\ 2{,}8111 \end{bmatrix}$$

Řešení

$$y_h = 2\varphi_0 + 2{,}2996\,\varphi_1 + 2{,}5732\,\varphi_2 + 2{,}8111\,\varphi_3 + 3\varphi_4$$