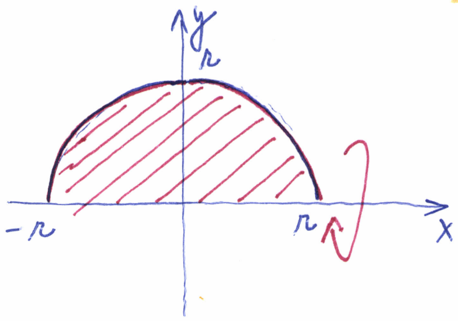
Objem & povrch koule



Explicitní rovnice

$$y = \sqrt{r^2 - x^2} = f(x)$$

$$x \in \langle -r, r \rangle$$

Parametrické rovnice

$$x = r\cos t = \varphi(t)$$

$$y = \underbrace{r\sin t}_{=\psi(t)}, \quad t \in \langle 0, \pi \rangle$$

Objem:

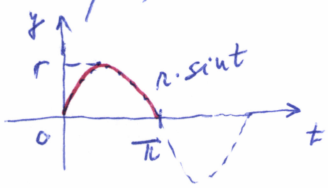
$$V = \pi \int_a^b f^2(x)\,dx = \pi \int_{-r}^{r} r^2 - x^2\,dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \pi \left(r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3} \right)$$

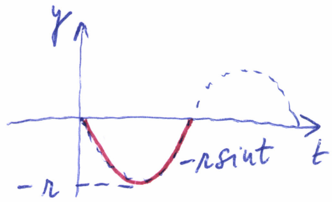
$$= \frac{4}{3}\pi r^3$$

$$V = \pi \int_\alpha^\beta \psi^2(t)\,|\varphi'(t)|\,dt, \quad \text{kde} \quad \psi(t) \geq 0 \quad \text{a} \quad \varphi'(t) \neq 0 \quad \forall t \in (\alpha, \beta)$$

$$\psi(t) = r\,\cancel{\cos t}\,\sin t \geq 0 \quad ✓$$

$$\varphi'(t) = -r\sin t \neq 0 \quad ✓$$





$$V = \pi \int_0^\pi r^2 \sin^2 t \cdot |-r\sin t|\,dt = \pi r^3 \int_0^\pi \sin^3 t\,dt = \left| \begin{array}{l} \cos t = s \\ -\sin t\,dt = ds \\ \begin{array}{c|c|c} t & 0 & \pi \\ \hline s & 1 & -1 \end{array} \end{array} \right|$$

$$= -\pi r^3 \int_{+1}^{-1} (1 - s^2)\,ds = \pi r^3 \left[s - \frac{s^3}{3} \right]_{-1}^{1} =$$

$$= \pi r^3 \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) = \frac{4}{3}\pi r^3$$

Povrch

$$P = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$f'(x) = \frac{1}{\cancel{2}} \cdot \frac{-\cancel{2}x}{\sqrt{r^2-x^2}} = -\frac{x}{\sqrt{r^2-x^2}}$$

$$P = 2\pi \int_{-r}^{r} \sqrt{r^2-x^2} \cdot \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = 2\pi \int_{-r}^{r} \sqrt{r^2-x^2} \cdot \sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx$$

$$= 2\pi \int_{-r}^{r} \cancel{\sqrt{r^2-x^2}} \cdot \frac{r}{\cancel{\sqrt{r^2-x^2}}}\,dx = 2\pi \left[r x \right]_{-r}^{r} = \underline{\underline{4\pi r^2}}$$

$$P = 2\pi \int_\alpha^\beta \psi(t)\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt \quad , \text{kde } \psi(t) \geq 0 \;\; \forall t \in (\alpha, \beta)$$

$\varphi'(t) = -r\sin t$

$\psi'(t) = r\cos t$

$$P = 2\pi \int_0^\pi r\cdot \sin t \cdot \sqrt{r^2\cdot \sin^2 t + r^2\cos^2 t}\,dt$$

$$= 2\pi \int_0^\pi r \sin t \; r\cdot \underbrace{\sqrt{\sin^2 t+\cos^2 t}}_{1}\,dt = 2\pi r^2 \int_0^\pi \sin t\,dt$$

$$= 2\pi r^2 \left[-\cos t \right]_0^\pi = \underline{\underline{4\pi r^2}}$$